

DEFORMATIONS OF ANNULI ON RIEMANN SURFACES WITH SMALLEST MEAN DISTORTION

DAVID KALAJ

ABSTRACT. Let A and A' be two circular annuli and let ρ be a radial metric defined in the annulus A' . Consider the class \mathcal{H}_ρ of ρ -harmonic mappings between A and A' . It is proved recently by Iwaniec, Kovalev and Onninen that, if $\rho = 1$ (i.e. if ρ is Euclidean metric) then \mathcal{H}_ρ is not empty if and only if there holds the Nitsche condition (and thus is proved the J. C. C. Nitsche conjecture). In this paper we formulate a condition (which we call ρ -Nitsche conjecture) which corresponds to \mathcal{H}_ρ and define ρ -Nitsche harmonic maps. We determine the extremal mappings with smallest mean distortion for mappings of annuli w.r. to the metric ρ . As a corollary, we find that ρ -Nitsche harmonic maps are Dirichlet minimizers among all homeomorphisms $h : A \rightarrow A'$. However, outside the ρ -Nitsche condition of the modulus of the annuli, within the class of homeomorphisms, no such energy minimizers exist. This extends some recent results of Astala, Iwaniec and Martin (ARMA, 2010) where it is considered the case $\rho = 1$ and $\rho = 1/|z|$.

1. INTRODUCTION

1.1. Mappings of finite distortion. A homeomorphism $w = f(z)$ between planar domains Ω and D has finite distortion if

- a) f lies in the Sobolev space $W_{loc}^{1,1}(\Omega, D)$ of functions whose first derivatives are locally integrable, and
- b) f satisfies the distortion inequality

$$|f_{\bar{z}}| \leq \mu(z)|f_z|,$$

$0 \leq \mu(z) < 1$ almost everywhere in Ω . Such mappings are generalizations of quasiconformal homeomorphisms where one works with the stronger assumption $\mu(z) \leq k < 1$. Mappings of finite distortion have found considerable interest in geometric function theory and the mathematical theory of elasticity. A comprehensive overview of the theory of mappings of finite distortion in two-dimensions can be found in [3]. The Jacobian determinant of a mapping f of finite distortion is non-negative almost everywhere, since

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = (1 - \mu(z)^2)|f_z|^2 \geq 0.$$

2000 *Mathematics Subject Classification.* 58E20, 30F45.

Key words and phrases. Harmonic maps, Riemann surfaces, Modulus of annuli, Finite distortion, Radial metric.

The distortion function of particular interest to us in this article is defined by the rule

$$(1.1) \quad K(z, f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{\|Df(z)\|^2}{J_f(z)}$$

if $J_f(z) > 0$. Here

$$\|A\|^2 = \frac{1}{2} \text{Tr}(A^T A)$$

is the square of mean Hilbert-Schmidt norm. We conveniently set $K(z, f) = 1$ if $f_z = f_{\bar{z}} = 0$. Notice that then $K(z, f) = 1$ and we have the equality $K(z, f) = 1$ if and only if f is conformal, by the Looman Menchoff theorem.

1.2. Radial metrics. By \mathbb{U} we denote the unit disk $\{z : |z| < 1\}$, by $\overline{\mathbb{C}}$ is denoted the extended complex plane. Let Σ be a Riemannian surface over a domain C of the complex plane or over $\overline{\mathbb{C}}$ and let $p : C \mapsto \Sigma$ be a universal covering. Let ρ_Σ be a conformal metric defined in the universal covering domain C or in some chart D of Σ . It is well-known that C can be one of three sets: \mathbb{U} , \mathbb{C} and $\overline{\mathbb{C}}$. Then the distance function is defined by

$$d(a, b) = \inf_{a, b \in \gamma} \int_0^1 \rho_\Sigma(\tilde{\gamma}(t)) |\tilde{\gamma}'(t)| dt,$$

where $\tilde{\gamma}, \tilde{\gamma}(0) = 0$, is a lift of γ , i.e. $p(\tilde{\gamma}(t)) = \gamma(t)$, $\gamma(0) = a$, $\gamma(1) = b$.

The Gauss curvature of the surface (and of the metric ρ_Σ) is given by

$$K = -\frac{\Delta \log \rho_\Sigma}{\rho_\Sigma^2}.$$

In this paper we will consider those surfaces Σ , whose metric have the form

$$\rho_\Sigma(z) = h(|z|^2),$$

defined in some chart $A' = \{z : \tau < |z| < \sigma\}$ of Σ (not necessarily in the whole universal covering surface). Here h is an positive twice differentiable function. We call these metrics *radial symmetric*.

Definition 1.1. The radial metric ρ is called a *regular metric* if

$$\inf_{\tau < s < \sigma} s\rho(s) = \lim_{s \rightarrow \tau+0} s\rho(s)$$

and has bounded Gauss curvature K .

Euclidean metric $\rho(z) = 1$, and the metric $\rho(z) = \frac{1}{|z|}$ are regular metrics which are considered by Astala, Iwaniec and Martin in the paper [2]. The authors settled corresponding problems of deformations of annuli with smallest mean distortion with respect to these two metrics. The aim of this paper is to extend the results in [2] for all regular metrics (see Theorem 3.1, Theorem 3.3, Corollary 3.6 and Corollary 3.4).

In Appendix below it is showed that, most of known metrics with constant Gauss curvatures defined in annuli are regular metrics. Such metrics are for example, hyperbolic metric

$$\lambda(z) = \frac{2}{1 - |z|^2}, \quad K = -1$$

defined in the unit disk \mathbb{U} and Riemann metric

$$\eta(z) = \frac{2}{1 + |z|^2}, \quad K = 1$$

defined in the Riemann sphere $S^2 := \overline{\mathbb{C}}$.

1.3. Harmonic mappings between Riemann surfaces. Let (M, σ) and (N, ρ) be Riemann surfaces with metrics σ and ρ , respectively. If a mapping $f : (M, \sigma) \rightarrow (N, \rho)$ is C^2 , then f is said to be harmonic (to avoid the confusion we will sometimes say ρ -harmonic) if

$$(1.2) \quad f_{z\bar{z}} + (\log \rho^2)_w \circ f f_z f_{\bar{z}} = 0,$$

where z and w are the local parameters on M and N respectively. Also f satisfies (1.2) if and only if its Hopf differential

$$(1.3) \quad \Psi = \rho^2 \circ f f_z \overline{f_z}$$

is a holomorphic quadratic differential on M .

For $g : M \mapsto N$ the energy integral is defined by

$$(1.4) \quad E_\rho[g] = \int_M (|\partial g|^2 + |\bar{\partial} g|^2) dV_\sigma,$$

where ∂g , and $\bar{\partial} g$ are the partial derivatives taken with respect to the metrics ρ and σ , and dV_σ is the volume element on (M, σ) . Assume that energy integral of f is bounded. Then f is harmonic if and only if f is a critical point of the corresponding functional where the homotopy class of f is the range of this functional. For this definition and the basic properties of harmonic maps see [24].

It follows from the definition that,

Lemma 1.2. *If a is holomorphic and f is harmonic, then $f \circ a$ is harmonic.*

Using the fact that the function defined in (1.3) is holomorphic, the following well known lemma can be proved (see e.g. [19]).

Lemma 1.3. *Let (S_1, ρ_1) and (S_2, ρ_2) and (R, ρ) be three Riemann surfaces. Let g be an isometric transformation of the surface S_1 onto the surface S_2 :*

$$\rho_1^2(\omega) |d\omega|^2 = \rho_2^2(w) |dw|^2, \quad w = g(\omega).$$

Then $f : R \mapsto S_1$ is ρ_1 -harmonic if and only if $g \circ f : R \mapsto S_2$ is ρ_2 -harmonic. In particular, if g is an isometric self-mapping of S_1 , then f is ρ_1 -harmonic if and only if $g \circ f$ is ρ_1 -harmonic.

Example 1.4. Let ρ be the Riemann metric

$$\rho = \frac{2}{1 + |z|^2}.$$

Equation (1.2) becomes

$$(1.5) \quad u_{z\bar{z}} - \frac{2\bar{u}}{1 + |u|^2} u_z \cdot u_{\bar{z}} = 0.$$

Notice this important example. The Gauss map of a surface Σ in \mathbb{R}^3 sends a point on the surface to the corresponding unit normal vector $\mathbf{n} \in \overline{\mathbb{C}} \cong S^2$. In terms of a conformal coordinate z on the surface, if the surface has *constant mean curvature*, its Gauss map $\mathbf{n} : \Sigma \mapsto \overline{\mathbb{C}}$, is a Riemann harmonic map [23].

Example 1.5. If $u : \mathbb{U} \mapsto \mathbb{U}$ is a harmonic mapping with respect to the hyperbolic metric

$$\lambda = \frac{2}{1 - |z|^2}$$

then Euler-Lagrange equation of u is

$$(1.6) \quad u_{z\bar{z}} + \frac{2\bar{u}}{1 - |u|^2} u_z \cdot u_{\bar{z}} = 0.$$

An important example of hyperbolic harmonic mapping is the Gauss map of a space-like surfaces with constant mean curvature H in the Minkowski 3-space $M^{2,1}$ (see [7], [21] and [25]).

2. RADIAL ρ -HARMONIC MAPPINGS AND ρ -NITSCHKE CONJECTURE

The conjecture in question concerns the existence of a harmonic homeomorphism between circular annuli $A(r, 1)$ and $A(\tau, \sigma)$, and is motivated in part by the existence problem for doubly-connected minimal surfaces with prescribed boundary. In 1962 Nitsche [22] observed that the image annulus cannot be too thin, but it can be arbitrarily thick (even a punctured disk). Then he conjectured that for such a mapping to exist we must have the following inequality, now known as the Nitsche bound:

$$\frac{\sigma}{\tau} \geq \frac{1}{2} \left(\frac{1}{r} + r \right).$$

For some results concerning the partial solution of Nitsche conjecture see papers [18], [26] and [20]. For the generalization of this conjecture to \mathbb{R}^n and some related results we refer to [17]. For the case of hyperbolic harmonic mappings we refer to [8]. Some other generalization has been done in [18] (see Proposition 2.3 below). The Nitsche conjecture for Euclidean harmonic mappings is settled recently in [12] by Iwaniec, Kovalev and Onninen, showing that, only radial harmonic mappings

$$h(\zeta) = C \left(\zeta - \frac{\omega}{\bar{\zeta}} \right),$$

$C \in \mathbb{C}$, $\omega \in \mathbb{R}$, $|C|(1 - \omega) = \sigma$, which inspired the Nitsche conjecture, make the extremal distortion of rounded annuli.

In this section, we will state a similar conjecture with respect to ρ -harmonic mappings. In order to this, we will find examples of radial ρ -harmonic maps between annuli. We put

$$w(z) = g(s)e^{it}, \quad z = se^{it}$$

where g is a increasing or a decreasing function to be chosen. This will include all radial harmonic mappings.

Direct calculations yield

$$(2.1) \quad w_{z\bar{z}} = \frac{1}{4}\Delta w = \frac{1}{4s^2} (s^2 w_{ss} + s w_s + w_{tt})$$

and

$$(2.2) \quad w_z w_{\bar{z}} = \frac{1}{4s^2} (s^2 w_s^2 - w_t^2).$$

Inserting this into harmonic equation (1.2), we obtain

$$s^2 g'' + s g' - g + \frac{2\rho_w}{\rho} (s^2 g'^2 - g^2) = 0.$$

Let $s = e^x$ and

$$(2.3) \quad \varrho = \frac{1}{\rho}.$$

Put $y(x) = g(e^x)$. Then the corresponding differential equation is

$$y'' - y = \frac{\varrho'(y)}{\varrho(y)} (y'^2 - y^2).$$

After some changes we obtain that, the general solution to this equation is

$$x + c_1 = \int \frac{dy}{\sqrt{y^2 + c\varrho^2}},$$

where c and c_1 are certain constants. The mapping w given by

$$(2.4) \quad w(se^{it}) = q^{-1}(s)e^{it},$$

where

$$(2.5) \quad q(s) = \exp(\varphi(s)) = \exp \left(\int_{\sigma}^s \frac{dy}{\sqrt{y^2 + c\varrho^2}} \right), \quad \tau \leq s \leq \sigma,$$

and c satisfies the condition:

$$(2.6) \quad y^2 + c\varrho^2(y) \geq 0, \quad \text{for } \tau \leq s \leq \sigma,$$

is a ρ -harmonic mapping between annuli $A = A(r, 1)$ and $A' = A(\tau, \sigma)$, where

$$(2.7) \quad r = \exp \left(\int_{\sigma}^{\tau} \frac{dy}{\sqrt{y^2 + c\varrho^2}} \right).$$

The harmonic mapping w is normalized by

$$w(e^{it}) = \sigma e^{it}.$$

The mapping $w = h^c(z)$ is a diffeomorphism, and we will call it ρ -Nitsche mapp. From now on we will assume that the metric ρ is regular in the sense of Definition 1.1. Then (2.6) is equivalent to

$$(2.8) \quad \tau^2 + c\rho^2(\tau) \geq 0.$$

Accordingly, for $c = -\tau^2\rho^2(\tau)$, we have well defined function

$$q^\#(s) = \exp \left(\int_\sigma^s \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\rho^2}} \right), \tau \leq s \leq \sigma.$$

The mapping $h^\# : A \rightarrow A'$ defined by $h^\#(se^{it}) = (q^\#)^{-1}(s)e^{it}$ is called the *critical Nitsche map*.

For $\tau \leq s \leq \sigma \leq 1$ we have:

$$(2.9) \quad s^2\varphi'(s)^2 - 1 = \frac{-c}{s^2\rho^2 + c} \begin{cases} \leq 0, & \text{if } c \geq 0; \\ \geq 0, & \text{if } -\tau^2\rho^2(\tau) \leq c \leq 0. \end{cases}$$

Notice that, the mapping

$$f^c(se^{it}) = q(s)e^{it} : A \rightarrow A'$$

is the inverse of the harmonic diffeomorphism w .

Conjecture 2.1. *Let ρ be a regular metric. If $r < 1$, and there exists a ρ -harmonic mapping of the annulus $A' = A(r, 1)$ onto the annulus $A = A(\tau, \sigma)$, then*

$$(2.10) \quad r \geq \exp \left(\int_\sigma^\tau \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - \tau^2\rho^2(\tau)}} \right).$$

Notice that if $\rho = 1$, then this conjecture coincides with standard Nitsche conjecture.

The following example assert that, in the settings of the previous conjecture a upper bound for r do not holds. In other words the image domain it can be arbitrarily thick (even a punctured disk). This differs harmonic mappings from conformal mappings and quasiconformal mappings.

Example 2.2. Let ρ be a metric defined on the unit disk \mathbb{U} , and take $A' = A(0, 1)$ ($0 = \tau < \sigma = 1$). Let $c > 0$ and

$$q(s) = \exp \left(\int_1^s \frac{\rho(s)ds}{\sqrt{c + s^2\rho^2(s)}} \right).$$

Define $w(z) = q^{-1}(s)e^{it}$. Then w is a ρ -harmonic diffeomorphism between annuli $A(r_c, 1)$ and the degenerated annuli $A(0, 1)$. Here

$$0 < r_c = \exp \left(- \int_0^1 \frac{\rho(s)ds}{\sqrt{c + s^2\rho^2(s)}} \right) < 1$$

and

$$\lim_{c \rightarrow +\infty} r_c = 1.$$

We call the inequality (2.10) ρ -Nitsche condition. The converse inequality

$$(2.11) \quad r < \exp \left(\int_{\sigma}^{\tau} \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - \tau^2\rho^2(\tau)}} \right),$$

we will call *the fatness condition*.

If $r < 1$ satisfies the condition (2.10), then by continuity argument, there exists an c satisfying

$$c \geq -\tau^2\rho^2(\tau)$$

such that

$$(2.12) \quad r = \exp \left(\int_{\sigma}^{\tau} \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) + c}} \right).$$

The following theorem is a partial result in solving the previous conjecture

Proposition 2.3. [16] *Assume that ρ is a metric defined in the disk $\{z : |z| \leq \sigma\}$ with positive or negative Gauss curvature $K(z)$ and let*

$$h(x) = \int_0^x \rho(t)dt, \quad 0 \leq x \leq \sigma.$$

If there exists a ρ -harmonic diffeomorphism between the annuli $A = A(r, 1) = \{z \in \mathbb{C} : r < |z| < 1\}$ and $A' = \{z \in \mathbb{C} : \tau < |z| < \sigma\}$, then

$$(2.13) \quad \frac{h(\sigma)}{h(\tau)} \geq 1 + \frac{\tau}{2h(\tau)} \log^2 r \begin{cases} (t\rho(t))'|_{t=\tau}, & \text{if } K \text{ is negative;} \\ (t\rho(t))'|_{t=\sigma}, & \text{if } K \text{ is positive.} \end{cases}$$

3. STATEMENT OF THE MAIN RESULTS

The classical formulations of the extremal Grötsch and Teichmüller problems are concerned with finding mappings $\Omega \rightarrow D$ in some class (for instance, with free or prescribed boundary values) which have smallest L^∞ -norm of the distortion function, thus "extremal quasiconformal mappings". In this article we shall investigate mappings in some class which minimize integral means with respect to appropriate metrics of the distortion function $\mathbb{K}(z, f)$. The case of bounded simply connected domains, without boundary data, is trivial; the extremals are the conformal mappings of Ω onto D asserted to exist by the Riemann mapping theorem (the simply connected case where the boundary data is prescribed is solved in [4]). For the free boundary problem, Astala, Iwaniec and Martin [2] considered the first nontrivial case where there are conformal invariants; namely doubly connected domains and, in particular, annuli. Given two annuli

$$A' = \{w : \tau < |w| < \sigma\}, \quad A = \{z : r < |z| < 1\},$$

they consider homeomorphisms of finite distortion $f : A' \rightarrow A$ with respect to the Euclidean metric and the metric $\rho = 1/|z|$. We shall consider the same problem but for an arbitrary *regular metric*. Here note that $|f|$ extends continuously to $\overline{A'}$, with values r and 1 on the boundary of A . We shall normalize our mappings in the obvious way so that

$$|f(z)| = r \text{ for } |z| = \tau \text{ and } |f(z)| = \sigma \text{ for } |z| = 1$$

Let $\mathcal{F} = \mathcal{F}(A', A)$ denote the family of all normalized homeomorphisms $f : A' \rightarrow A$ of finite distortion. Since A' and A are certainly diffeomorphic $\mathcal{F} \neq \emptyset$.

Let $z = x + iy = se^{it}$ and

$$dm(z) = dx dy = s ds dt$$

be the usual *Lebesgue measure* on the complex plane \mathbb{C} . The integral mean of the distortion function $K(z, f)$ which concern us in this work is

$$\mathcal{K}_\rho[f] = \int_{A'} \mathbb{K}(z, f) \rho^2(z) dm(z),$$

where ρ is a given *regular metric*. The minimization problem we address here is to evaluate the following infimum

$$(3.1) \quad \inf \{ \mathcal{K}_\rho[f] : f \in \mathcal{F}(A', A) \}.$$

Further, we should decide if the infimum is attained and, in that case, prove uniqueness (up to the obvious rotational symmetry of the annuli). The concept of the conformal modulus will prove useful in proving our results. It is convenient to take the following definition of the modulus of an annulus $A(p, q) := \{z : p < |z| < q\}$

$$(3.2) \quad \text{Mod}(A(p, q)) = 2\pi \log \frac{q}{p} = \int_{A(p, q)} \frac{dm(z)}{|z|^2}.$$

Note that, the standard definition of modulus is indeed $\text{mod}(A(p, q)) = \frac{1}{2\pi} \log \frac{q}{p}$. Every topological annulus R is conformally equivalent to a round annulus A , and we can set $\text{Mod}(R) = \text{Mod}(A)$.

The main theorems of this paper are the following:

Theorem 3.1. *Let ρ be a regular metric. Let A and A' be annuli satisfying the condition (2.10). Among all mappings $f \in \mathcal{F}(A, A')$ the infimum of*

$$(3.3) \quad \int_{A'} \mathbb{K}(z, f) \rho^2(z) dm(z)$$

is attained by the function

$$(3.4) \quad f^c(z) = \exp \left(i(t + \alpha) + \int_\sigma^s \frac{\rho(y) dy}{\sqrt{y^2 \rho^2(y) + c}} \right), \quad z = se^{it}, \alpha \in [0, 2\pi),$$

where c is given by (2.12). Its inverse h^c is ρ -harmonic between annuli A' and A .

Remark 3.2. In the special cases where $\rho(z) = 1$, we easily obtain that

$$f^c(z) = C(c) \frac{z}{|z|} \left(|z| + \sqrt{|z|^2 + c} \right)$$

which is the inverse of the Nitsche's map

$$h^c(\zeta) = C'(c) \left(\zeta - \frac{\omega(c)}{\bar{\zeta}} \right),$$

where $\omega(c)$ is a positive constant and $C(c), C'(c) \in \mathbb{C}$.

If $\rho(z) = |z|^{-1}$, and $z = se^{it}$, then f^c is a power function

$$f^c(z) = \sigma s^{-\alpha} |z|^{\alpha-1} z, \quad \text{where } \alpha = \alpha(c) = \frac{\text{Mod}(A')}{\text{Mod}(A)}.$$

See [2, Theorem 1 & Theorem 2] for the same conclusion.

Theorem 3.3. *Let ρ be a regular metric. Under the fatness condition (2.11), the infimum at (3.3) is not attained by any homeomorphism $f \in \mathcal{F}(A, A')$ (more generally, it is not attained by any continuous mapping of finite distortion of A onto A'). Moreover for the inverse of critical Nitsche map $f^\#(se^{it}) = e^{\varphi^\#(s)+it}$, where*

$$(3.5) \quad \varphi^\#(s) = \int_\sigma^s \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(\tau) \varrho^2(y)}}, \quad \tau \leq s \leq \sigma,$$

there holds the sharp inequality

$$(3.6) \quad \int_{A'} \mathbb{K}(z, f) \rho^2(s) dm(z) \geq \int_{A'} \mathbb{K}(z, f^\#) \rho^2(s) dm(z) + \frac{\tau^2 \rho^2(\tau)}{2} \text{Mod } A(r, r'),$$

where

$$r' = \exp \left(\int_\sigma^\tau \frac{\rho(y) dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right) \quad (r < r' < 1).$$

The minimization of the integral means of the distortion functions of homeomorphisms $f : A' \rightarrow A$ turns to be equivalent to the Dirichlet type problem for the inverse mapping $h = f^{-1} : A \rightarrow A'$. If a homeomorphism $f \in W_{loc}^{1,1}(A', A)$ has integrable distortion, then $h \in W^{1,2}(A, A')$ and we can consider the energy functional

$$(3.7) \quad E_\rho[h] = \int_A \|Dh(\zeta)\|^2 \rho^2(h(\zeta)) dm(\zeta) = \int_{A'} \mathbb{K}(z, f) \rho^2(z) dm(z).$$

In general, the converse is not true, because the inverse of a homeomorphism $h \in W^{1,2}(A, A')$ need not belong to the Sobolev class $W_{loc}^{1,1}(A, A')$. It has bounded variation but fails to be ACL (absolutely continuous on lines), see [10] for related results. As in [2] we prove the correction lemma and overcame this problem.

Accordingly, for every homeomorphism $h \in W^{1,2}(A, A')$, we can construct a homeomorphism $\tilde{h} \in W^{1,2}(A, A')$, with $E_\rho[\tilde{h}] \leq E_\rho[h]$, whose inverse lies in $\mathcal{F}(A', A)$. As a consequence, the minimization problem for $\mathcal{K}_\rho[f]$ is equivalent to the minimization problem for $E_\rho[h]$.

The energy integral defined in (1.4) coincides with the energy integral in (3.7). We should point out that, the image surface M is indeed the annulus A with metric ρ and make use of the formulas

$$|\partial g|^2 = \frac{\rho^2 \circ g |g_z|^2}{\sigma^2(z)}, \quad |\bar{\partial} g|^2 = \frac{\rho^2 \circ g |g_{\bar{z}}|^2}{\sigma^2(z)} \quad \text{and} \quad dV_\sigma = \sigma^2(z) dm(z).$$

From Theorem 3.1 and Lemma 5.1 we deduce

Corollary 3.4. *Let ρ be a regular metric. Within the Nitsche rang (2.10), for the annuli A and A' , the absolute minimum of the energy integral*

$$h \rightarrow E_\rho[h], \quad h \in W^{1,2}(A, A')$$

is attained by a ρ -Nitsche map

$$h^c(z) = q^{-1}(s)e^{i(t+\beta)}, \quad z = se^{it}, \quad \beta \in [0, 2\pi),$$

where

$$q(s) = \exp \left(\int_\sigma^s \frac{dy}{\sqrt{y^2 + c\rho^2}} \right), \quad \tau < s < \sigma.$$

Remark 3.5. If R is an doubly connected surface, conformal to a given annulus A , and (R', ρ') another annulus isometric to a given annulus (A', ρ) , then by Lemma 1.2 and Lemma 1.3, the minimization of the energy integral

$$h \rightarrow E_\rho[h], \quad h \in W^{1,2}(A, A')$$

is equivalent to the minimization of the energy integral

$$k \rightarrow E_{\rho'}[k], \quad k \in W^{1,2}(R, R').$$

Thus Corollary 3.4 can be formulated in terms of not-necessarily rounded annuli.

From Theorem 3.3 and Lemma 5.1 we deduce

Corollary 3.6. *Let ρ be a regular metric. Outside the ρ -Nitsche range (2.10) for the annuli A and A' , the infimum of the energy functional $E_\rho[h]$ is not attained by any homeomorphism $h' \in W^{1,2}(A, A')$.*

Remark 3.7. Theorem 3.1, Theorem 3.3, Corollary 3.6 and Corollary (3.4) are generalizations of corresponding [2, Thm 1, Thm 2, Thm 3, Cor 1, Cor 2 and Cor 3].

4. PROOF OF THEOREM 3.1 AND THEOREM 3.3

We need the following elementary formulas in the sequel. Let $z = se^{it}$. Then

$$(4.1) \quad |f_z|^2 + |f_{\bar{z}}|^2 = \frac{1}{2}(|f_s|^2 + s^{-2}|f_t|^2)$$

and

$$(4.2) \quad J_f(z) = \frac{1}{s} \Im(f_t \overline{f_s}).$$

From (4.1) and (4.2) we obtain

$$(4.3) \quad \mathbb{K}(z, f) = \frac{s|f_s|^2 + s^{-1}|f_t|^2}{2\Im(f_t \overline{f_s})}.$$

If a mapping f is radial stretching between annuli A' and A , then for some increasing function $P(s)$, $\tau < s < \sigma$ there holds the formula

$$f(se^{it}) = P(s)e^{it}.$$

If $\Phi(s) = \log P(s)$, then we can express the distortion function as

$$(4.4) \quad \mathbb{K}(z, f) = \frac{1}{2} \left(s\Phi'(s) + \frac{1}{s\Phi'(s)} \right).$$

Lemma 4.1. *Let f be a mapping of finite distortion and φ be a differentiable monotonic function. For $z \in A^+ := \{z : J_f(z) > 0\}$ we have the following equivalent inequalities*

$$(4.5) \quad \mathbb{K}(z, f) \geq s\varphi'(s) + \frac{1 - s^2(\varphi'(s))^2}{2s^2J_f(z)}|f_t|^2$$

and

$$(4.6) \quad \mathbb{K}(z, f) \geq \frac{1}{s\varphi'(s)} + \frac{s^2\varphi'(s)^2 - 1}{\varphi'(s)^2} \frac{1}{2s^2J_f(z)}|f_s|^2.$$

In both inequalities the equality is attained a.e. if and only if $f_s = -i\varphi'(s)f_t$ for $z \in A^+$.

Proof. In view of (4.3), it is easily to verify that, the following trivial inequality

$$(4.7) \quad \left| f_t - \frac{if_s}{\varphi'(s)} \right| \geq 0$$

is equivalent with both inequalities (4.5) and (4.6). \square

4.1. Proof of inequalities.

Lemma 4.2 (The main lemma). *Let ρ be a regular metric and let $f : A' \rightarrow A$ be a homomorphism of finite distortion between annuli $A' = A(\tau, \sigma)$ and $A = A(r, 1)$.*

a) *Assume that there holds the ρ -Nitsche condition (2.10) and let c be defined by (2.12). For $f^c(se^{it}) = e^{\varphi(s)+it}$, where*

$$\varphi(s) = \int_{\sigma}^s \frac{dy}{\sqrt{y^2 + c\rho^2(y)}}, \quad \tau \leq s \leq \sigma,$$

there holds the inequality

$$\int_{A'} \mathbb{K}(z, f)\rho^2(s)dm(z) \geq \int_{A'} \mathbb{K}(z, f^c)\rho^2(s)dm(z).$$

b) *Assume that there holds the fatness condition (2.11). Now we make use of inverse of critical Nitsche map. For $f^{\#}(se^{it}) = e^{\varphi^{\#}(s)+it}$, where*

$$\varphi^{\#}(s) = \int_{\sigma}^s \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\rho^2(y)}}, \quad \tau \leq s \leq \sigma,$$

there holds the inequality

$$\int_{A'} \mathbb{K}(z, f)\rho^2(s)dm(z) \geq \int_{A'} \mathbb{K}(z, f^{\#})\rho^2(s)dm(z) + \frac{\tau^2\rho^2(\tau)}{2}\text{Mod } A(r, r'),$$

where

$$r' = \exp \left(\int_{\sigma}^{\tau} \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right) \quad (r' > r).$$

Proof. Let $z = se^{it}$, $s = |z|$, $t \in [0, 2\pi)$ and $A^+ = \{z : J_f(z) > 0\}$. We will apply Lemma 4.1 with

$$\varphi(s) = \int_{\sigma}^s \frac{dy}{\sqrt{y^2 + c\varrho^2}}, \quad \tau \leq s \leq \sigma.$$

Proof of a). Since $\varrho = 1/\rho$, we have

$$1 - s^2 \varphi'(s)^2 = \frac{c}{s^2 \rho^2 + c}.$$

In view of this fact, we divide the proof into two cases.

• **The case $c \geq 0$.** Observe first that, for almost every $z = se^{it} \in A'$

$$(4.8) \quad s\varphi'(s) = \frac{s}{\sqrt{s^2 + c\varrho^2(s)}} \leq 1 \leq \mathbb{K}(z, f).$$

According to (4.5) and (4.8), we have

$$(4.9) \quad \begin{aligned} \int_{A'} \mathbb{K}(z, f) \rho^2(s) &\geq \int_{A' \setminus A^+} s \rho^2(s) \varphi'(s) + \int_{A^+} s \rho^2(s) \varphi'(s) \\ &+ \int_{A^+} \rho^2(s) \frac{1 - s^2(\varphi'(s))^2}{2s^2 J_f(z)} |f_t|^2 \\ &= \int_{A'} s \rho^2(s) \varphi'(s) + \int_{A^+} \frac{c}{2s^2(s^2 + c\varrho^2(s))} \frac{|f_s|^2}{J_f(z)}. \end{aligned}$$

By Hölder inequality we obtain

$$(4.10) \quad \int_{A^+} \frac{|f_t|}{s|f|\sqrt{2(s^2 + c\varrho^2(s))}}} \leq \left(\int_{A^+} \frac{1}{2s^2(s^2 + c\varrho^2(s))} \frac{|f_t|^2}{J_f(z)} \right)^{1/2} \left(\int_{A^+} \frac{J_f(z)}{|f|^2} \right)^{1/2}.$$

By [2, Lemma 1] it follows that

$$(4.11) \quad \int_{A^+} \frac{J_f(z)}{|f|^2} \leq \int_{A'} \frac{J_f(z)}{|f|^2} \leq \text{Mod}(A).$$

Since

$$\int_0^{2\pi} \frac{|f_t(se^{it})|}{|f(se^{it})|} \geq \left| \int_0^{2\pi} \frac{f_t(se^{it})}{f(se^{it})} dt \right| = 2\pi,$$

from (4.11), we obtain

$$\begin{aligned} \int_{A'} \frac{1}{2s^2(s^2 + c\varrho^2(s))} \frac{|f_t|^2}{J_f(z)} &\geq \frac{1}{\text{Mod}(A)} \left(\int_{A'} \frac{|f_t|}{s|f|\sqrt{2(s^2 + c\varrho^2(s))}}} \right)^2 \\ &\geq \frac{4\pi^2}{\text{Mod}(A)} \left(\int_{\tau}^{\sigma} \frac{ds}{\sqrt{2(s^2 + c\varrho^2(s))}}} \right)^2. \end{aligned}$$

On the other hand,

$$\int_{A'} s\rho^2(s)\varphi'(s)dm(z) = 2\pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds.$$

Therefore

$$\begin{aligned} \int_{A'} \mathbb{K}(z, f)\rho^2(s)dm(z) &\geq 2\pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds \\ &\quad + \frac{4\pi^2}{\text{Mod}(A)} \left(\int_{\tau}^{\sigma} \frac{ds}{\sqrt{2(s^2/c + \rho^2(s))}} \right)^2. \end{aligned}$$

Since

$$(4.12) \quad \text{Mod}(A) = 2\pi \log \frac{1}{r} = 2\pi \int_{\tau}^{\sigma} \frac{ds}{\sqrt{s^2 + c\rho^2(s)}},$$

it follows that

$$\begin{aligned} \int_{A'} \mathbb{K}(z, f)\rho^2(s)dm(z) &\geq 2\pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds + \pi c \int_{\tau}^{\sigma} \frac{ds}{\sqrt{s^2 + c\rho^2(s)}} \\ &= \pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds \\ &\quad + \pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds + \pi c \int_{\tau}^{\sigma} \frac{ds}{\sqrt{s^2 + c\rho^2(s)}} \\ &= \pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{\sqrt{s^2 + c\rho^2(s)}} ds + \pi \int_{\tau}^{\sigma} \rho^2(s) \sqrt{s^2 + c\rho^2(s)} ds. \end{aligned}$$

For $f^c(se^{it}) = e^{\varphi(s)+it}$, where

$$\varphi(s) = \int_{\sigma}^s \frac{dy}{\sqrt{y^2 + c\rho^2}}, \quad \tau \leq s \leq \sigma,$$

by making use of formula (4.4), we have

$$\begin{aligned} \int_{A'} \mathbb{K}(z, f^c)\rho^2(s)dm(z) &= \frac{2\pi}{2} \int_{\tau}^{\sigma} s\rho^2(s) \left(\varphi'(s)s + \frac{1}{s\varphi'(s)} \right) ds \\ &= \pi \int_{\tau}^{\sigma} s\rho^2(s) \left(\frac{s}{\sqrt{s^2 + c\rho^2(s)}} + \frac{\sqrt{s^2 + c\rho^2(s)}}{s} \right) ds. \end{aligned}$$

Thus

$$\int_{A'} \mathbb{K}(z, f)\rho^2(s)dm(z) \geq \int_{A'} \mathbb{K}(z, f^c)\rho^2(s)dm(z).$$

• **The case** $-\tau^2\rho^2(\tau) \leq c \leq 0$. In this case we make use of (4.6). Observe also that, for almost every $z \in A'$

$$(4.13) \quad \frac{1}{s\varphi'(s)} = \frac{\sqrt{s^2 + c\rho^2(s)}}{s} \leq 1 \leq \mathbb{K}(z, f).$$

For $z \in A^+$, we obtain

$$\mathbb{K}(z, f)\rho^2(s) \geq \frac{\rho^2(s)}{s\varphi'(s)} + \frac{-c\rho^2(s)}{2s^2\rho^2(s)} \frac{|f_s|^2}{J_f}.$$

Hence by (4.13)

$$(4.14) \quad \begin{aligned} \int_{A'} \mathbb{K}(z, f)\rho^2(s) &\geq \int_{A' \setminus A^+} \frac{\rho^2(s)}{s\varphi'(s)} + \int_{A^+} \frac{\rho^2(s)}{s\varphi'(s)} + \int_{A^+} \frac{-c}{2s^2} \frac{|f_s|^2}{J_f} \\ &= \int_{A'} \frac{\rho^2(s)}{s\varphi'(s)} + \int_{A^+} \frac{-c}{2s^2} \frac{|f_s|^2}{J_f}. \end{aligned}$$

By [2, Lemma 3] and (4.12) we have

$$(4.15) \quad \int_{A^+} \frac{1}{s^2} \frac{|f_s|^2}{J_f} \geq \text{Mod}(A) = 2\pi \int_{\tau}^{\sigma} \frac{ds}{\sqrt{s^2 + c\varrho^2(s)}}.$$

On the other hand

$$(4.16) \quad \int_{A'} \frac{\rho^2(s)}{s\varphi'(s)} = 2\pi \int_{\tau}^{\sigma} \rho^2(s) \sqrt{s^2 + c\varrho^2(s)} ds.$$

Let

$$k(s) := \sqrt{s^2 + c\varrho^2(s)}.$$

From (4.14), (4.15) and (4.16) we obtain

$$(4.17) \quad \begin{aligned} \int_{A'} \mathbb{K}(z, f)\rho^2(s) &\geq -c\pi \int_{\tau}^{\sigma} \frac{ds}{k(s)} + 2\pi \int_{\tau}^{\sigma} \rho^2(s)k(s)ds \\ &= \pi \int_{\tau}^{\sigma} \rho^2(s)k(s)ds + \pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{k(s)}ds \\ &\quad - c\pi \int_{\tau}^{\sigma} \frac{ds}{k(s)} + \pi \int_{\tau}^{\sigma} \rho^2(s)k(s)ds - \pi \int_{\tau}^{\sigma} \frac{s^2\rho^2(s)}{k(s)}ds \\ &= \int_{A'} \mathbb{K}(z, f^c)\rho^2(s)dm(z) + X, \end{aligned}$$

where

$$X = \pi \int_{\tau}^{\sigma} \left(-\frac{c}{k(s)} + \rho^2(s)k(s) - \frac{s^2\rho^2(s)}{k(s)} \right) ds = 0.$$

This concludes the proof of a).

The proof of b). We proceed similarly as in the proof of a). For $\varphi = \varphi^{\#}$, according to (4.6), for $z \in A^+$ we have

$$\mathbb{K}(z, f)\rho^2(s) \geq \frac{\rho^2(s)}{s\varphi'(s)} + \frac{\tau^2\rho^2(\tau)}{2s^2} \frac{|f_s|^2}{J_f}.$$

In this case, for $c^{\#} = -\tau^2\rho^2(\tau)$, instead of (4.15) we have

$$(4.18) \quad \int_{A^+} \frac{1}{s^2} \frac{|f_s|^2}{J_f} \geq \text{Mod}(A) = 2\pi \log \frac{1}{r} > 2\pi \int_{\tau}^{\sigma} \frac{ds}{\sqrt{s^2 + c^{\#}\varrho^2(s)}}.$$

On the other hand, the relation (4.16) holds. Hence

$$(4.19) \quad \int_{A'} \mathbb{K}(z, f) \rho^2(s) dm(z) \geq \int_{A'} \mathbb{K}(z, f^\#) \rho^2(s) dm(z) + Y,$$

where

$$Y = \frac{-c^\#}{2} \left(2\pi \log \frac{1}{r} - 2\pi \log \frac{1}{r'} \right) = \frac{\tau^2 \rho^2(\tau)}{2} \text{Mod } A(r, r').$$

This completes the proof of the lemma. \square

Proof of Theorem 3.1. In view of Lemma 4.2, a), it remains to prove the equality statement. By [2, Lemma 3], the equality in (4.15) occurs if and only if almost everywhere on $A^+ := \{z \in A' : J_f(z) > 0\}$ we have

$$(4.20) \quad \Re \left(\frac{f_s}{f} \right) = \left| \frac{f_s}{f} \right|$$

and

$$(4.21) \quad \Im \left(\frac{f_t}{f} \right) = k$$

for some constant $k > 0$. On other hand, by Lemma 4.1 we have the equality

$$(4.22) \quad f_s = -i\varphi'(s)f_t,$$

for $z \in A^+$. Combining (4.20), (4.21) and (4.22) we arrive at the following system of PDE's

$$(4.23) \quad \frac{f_t}{f} = ik \quad \text{and} \quad \frac{f_s}{f} = k\varphi'(s).$$

Integrating the first equality over the unit circle gives $k = 1$. Let $g(s, t) = \log f(se^{it})$. Then g is well defined function from $[\tau, \sigma] \times [0, 2\pi]$ onto $[\log r, 0] \times [0, 2\pi]$. Then from (4.23) we obtain

$$g = it + \phi(s) \quad \text{and} \quad g = \varphi(s) + \psi(t),$$

for some functions ϕ and ψ . We obtain that, the general solution of system of PDE's (4.23) is

$$f(se^{it}) = Ce^{\varphi(s)+it}.$$

\square

Proof of Theorem 3.3. The second part of Theorem 3.3 follows by Lemma 4.2, b). It remains to show that, the inequality is sharp and it is not attained by any homeomorphism. Otherwise, if $f \in \mathcal{F}(A', A)$ would satisfy (3.5), f has to be of the form

$$f(se^{it}) = C \exp \left(it + \int_\sigma^s \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(\tau) \varrho^2(y)}} \right), \quad \tau \leq s \leq \sigma.$$

But then $r' = r$ which is a contradiction with (2.11). To show that the inequality is sharp we construct the minimizing sequence.

4.2. The minimizing sequence. We are now given two round annuli A and A' with inner and outer radii $r, 1$ and τ, σ respectively. Moreover we assume that the *fatness condition* (2.11) is satisfied. To show that the inequality (3.6) is sharp we construct a family $f_n : A' \rightarrow A$ of mappings of finite distortion. Let

$$r' = \exp \left(\int_{\sigma}^{\tau} \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - \tau^2\rho^2(\tau)}} \right).$$

Then $r < r' < 1$. Let $z = se^{it}$ and $n \in \mathbb{N}$ such that

$$1 - r \left(\frac{\sigma}{\tau} \right)^n < 0.$$

Define

$$f_n(z) = \begin{cases} \exp \left(it + \int_{\sigma}^s \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\varrho^2(y)}} \right) & \text{if } s_n \leq s \leq \sigma, \\ rz|z|^{n-1}/\tau^n & \text{if } \tau \leq s \leq s_n, \end{cases}$$

where s_n is a solution of the equation

$$(4.24) \quad \exp \left(\int_{\sigma}^{s_n} \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\varrho^2(y)}} \right) = r \left(\frac{s_n}{\tau} \right)^n.$$

To show that $s_n : \tau < s_n < \sigma$ exists satisfying the condition (4.24) take

$$p(s) = \exp \left(\int_{\sigma}^s \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\varrho^2(y)}} \right) - r \left(\frac{s}{\tau} \right)^n.$$

Then

$$p(\tau)p(\sigma) = (r' - r) \left(1 - r \left(\frac{\sigma}{\tau} \right)^n \right) < 0,$$

if n is big enough. From (4.24) we easily get the relation

$$(4.25) \quad \lim_{n \rightarrow \infty} n(s_n - \tau) = \tau \log \frac{r'}{r}.$$

Further, by a similar analysis as in [2, Section 9] we obtain that

$$\mathbb{K}(z, f_n) = \begin{cases} \left(\frac{s}{\sqrt{s^2 - \tau^2\rho^2(\tau)\varrho^2(s)}} + \frac{\sqrt{s^2 - \tau^2\rho^2(\tau)\varrho^2(s)}}{s} \right) & \text{if } s_n \leq s \leq \sigma, \\ \frac{1}{2} \left(n + \frac{1}{n} \right) & \text{if } \tau \leq s \leq s_n. \end{cases}$$

Finally by (4.25)

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{A'(\tau, \sigma)} \mathbb{K}(z, f_n) \rho^2(s) &= \int_{A'(\tau, \sigma)} \mathbb{K}(z, f^\#) \rho^2(s) + \lim_{n \rightarrow \infty} \int_{A'(\tau, s_n)} \mathbb{K}(z, f_n) \rho^2(s) \\ &= \int_{A'} \mathbb{K}(z, f^\#) \rho^2(s) + 2\pi \lim_{n \rightarrow \infty} \frac{n}{2} \int_{\tau}^{s_n} \rho^2(s) ds \\ &= \int_{A'} \mathbb{K}(z, f^\#) \rho^2(s) + \pi \rho^2(\tau) \lim_{n \rightarrow \infty} \frac{n}{2} (s_n^2 - \tau^2) \\ &= \int_{A'} \mathbb{K}(z, f^\#) \rho^2(s) + \frac{\tau^2 \rho^2(\tau)}{2} \text{Mod } A(r, r'). \end{aligned}$$

This finishes the proof of Theorem 3.3. \square

5. PROOF OF COROLLARIES 3.6 AND 3.4

In order to apply Theorems 3.1 and 3.3 to energy minimization problems for the inverse mappings (see (3.7)), proving in this way Corollaries 3.6 and 3.4, we need to establish the following

Lemma 5.1 (The correction lemma). *Let D be doubly connected domain in \mathbb{C} and let $A' = \{z : \tau < |z| < \sigma\}$. Let $h : D \rightarrow A'$, be a homeomorphism of finite Dirichlet energy,*

$$E_\rho[h] = \int_D \rho^2 \circ h (|h_z|^2 + |h_{\bar{z}}|^2) dm(z) < \infty.$$

Assume that ρ is a radial metric with bounded Gauss curvature satisfying

$$(5.1) \quad 0 < \inf_{w \in A'} \rho^2(w) \leq \sup_{w \in A'} \rho^2(w) < \infty.$$

Then there exists a homeomorphism $\tilde{h} : D \rightarrow A'$ such that

$$E_\rho[\tilde{h}] \leq E_\rho[h].$$

The inverse $\tilde{f} = \tilde{h}^{-1}$ belongs to $W^{1,1}(A', D)$ and has finite distortion. We have the identity

$$\mathcal{K}_\rho[\tilde{f}] = E_\rho[\tilde{h}].$$

Proof. We proceed as in [2, Lemma 7], however due to the different intrinsic geometry, we should provide a different proof. A domain G is said to be convex with respect to the metric ρ , if for $z, w \in G$, there exists a geodesic line $l[z, w] \subset G$, $z, w \in l[z, w]$ with respect to the metric ρ . By following an argument of Buser ([5, Pages 116-121], see also [6, The proof of Theorem 1]) there exists a triangulation of the Riemann surface (A', ρ) by a locally finite set of triangles Δ_j , $j = 1, \dots, m$ with pairwise disjoint interiors, such that the edges of triangles are geodesic arcs or part of boundary (such domains are lipschitz and convex domains (if the diameter is small enough)). Also we can assume that

$$\text{diam}_\rho(\Delta_j) := \sup\{d_\rho(z, w) : z, w \in \Delta_j\} < \varepsilon$$

where ε is a positive constant to be chosen later.

Let us sketch the construction of these geodesic triangles. Let \mathcal{P} be a finite set of points in A such that any two points of \mathcal{P} lie at ρ -distance $\leq \varepsilon$ and $\geq \varepsilon/2$ from each other.

Such a set \mathcal{P} is easily obtained by successively marking points in $\overline{A'}$ at pairwise distances $\geq \varepsilon/2$ until there is no more room for such points. The set \mathcal{P} is finite because the ρ -diameter of A' is finite. We add to the set \mathcal{P} a maximal finite set of boundary points at pairwise distance between $\varepsilon/2$ and ε . For arbitrary triple of points a, b, c from \mathcal{P} we can construct a (possible degenerated) triangle Δ_i , $i = 1, \dots, n$, $n = \binom{|\mathcal{P}|}{3}$ defined as follows. If a, b, c are inside A' then the wedges are geodesic. If for example a, b belong to the same connected component of $\partial A'$, then the wedge ab is the smaller boundary arc bounded by a and b . We exclude all

degenerated triangles, those who contain more than three points inside and those whose diameter is larger than ε . Thus we get the family Δ_j , $j = 1, \dots, m$. Then each Δ_j is contained in a geodesic disk $B_\varepsilon(p)$ with center at $p \in \Delta_j$ and radius ε .

Through the homeomorphism h we have a decomposition

$$D = \bigcup_{k=1}^m \overline{D_k},$$

where $\overline{D_k} = h^{-1}(\overline{\Delta_k})$

Let the Gauss curvature of the metric ρ satisfies $K \leq \kappa^2$. Let \mathcal{C} be a positive constant, smaller or equal to the minimal distance of a given point $p \in A'$ from its cut-locus. Because the metric is radial, the cut-locus of a point $p \in A'$ is the arc $[-\tau p, -\sigma p]$. Since the metric satisfies the condition (5.1) it follows that

$$\mathcal{C} = \inf\{d_\rho(x, -y) : \tau \leq x, y \leq \sigma\} > 0.$$

Take

$$(5.2) \quad \varepsilon < \min\left\{\frac{\pi}{2\kappa}, \mathcal{C}\right\}.$$

Consider the Dirichlet problem of finding a ρ -harmonic map $h_j : D_j \rightarrow A'$ with the given boundary values: $h_j|_{\partial D_j} = h$. Since $h_j(\partial D_j)$ is contained in a geodesic disk $B_\varepsilon(p)$ with:

- (1) radius $\varepsilon < \pi/(2\kappa)$;
- (2) the cut locus of the centre p disjoint from $B_\varepsilon(p)$,

by a result of Hildebrandt, Kaul and Widman [11] this Dirichlet problem has a solution contained in $B_\varepsilon(p)$.

Moreover by a result of Jost [15] we obtain that, since $h_j : \partial D_j \rightarrow A'$ is a homeomorphism onto a Lipschitz convex curve $\partial\Delta_j$, then the above solution h_j is a homeomorphism.

Let

$$\tilde{h} = \sum_{j=1}^m h_j(z) \chi_{\overline{D_j}}(z).$$

Then \tilde{h} is a homeomorphism by construction. By using the well known energy estimates we have

$$\int_{D_j} \|Dh_j\|^2 \rho^2(z) dm(z) \leq \int_{D_j} \|Dh\|^2 \rho^2(z) dm(z).$$

Let $\tilde{f} = \tilde{h}^{-1}$. Proceeding as in [2, Lemma 7], we obtain

$$\mathcal{K}_\rho[\tilde{f}] \leq E_\rho[h].$$

To continue observe that

$$E[h] = \int_D (|h_z|^2 + |h_{\bar{z}}|^2) dm(z) \leq \int_D \frac{\rho^2 \circ h}{\inf_{w \in A'} \rho^2(w)} (|h_z|^2 + |h_{\bar{z}}|^2) dm(z) < \infty.$$

By using a recent result of Hencl, Koskela and Onninen [9], we obtain that the homeomorphism $f : A' \rightarrow D$ of integrable distortion in the Sobolev class $W^{1,1}(A', D)$

has its inverse $\tilde{h} \in W^{1,2}(D, A')$. By introducing the change of variables $w = \tilde{h}(z)$, proceeding as in [9], by making use of formulas

$$\|B\| = \|B^T\|, (A^T)^{-1} \det A = \operatorname{adj} A, \|\operatorname{adj} A\| = \|A\|$$

we obtain the identity

$$\mathcal{K}_\rho[\tilde{f}] = E_\rho[\tilde{h}].$$

This completes the proof of the Lemma 5.1. \square

Remark 5.2. We think that some of the conditions of Lemma 5.1 concerning the metric ρ are superfluous.

6. APPENDIX

In this section we will give some examples of *regular* metrics.

6.1. Hyperbolic metrics. For every hyperbolic Riemann surface, the fundamental group is isomorphic to a Fuchsian group, and thus the surface can be modeled by a Fuchsian model \mathbb{U}/Γ , where \mathbb{U} is the unit disk and Γ is the Fuchsian group ([1]). If Ω is a hyperbolic region in the Riemann sphere $\overline{\mathbb{C}}$; i.e., Ω is open and connected with its complement $\Omega^c := \overline{\mathbb{C}} \setminus \Omega$ possessing at least three points. Each such Ω carries a unique maximal constant curvature -1 conformal metric $\lambda|dz| = \lambda_\Omega(z)|dz|$ referred to as the Poincaré hyperbolic metric in Ω . The domain monotonicity property, that larger regions have smaller metrics, is a direct consequence of Schwarz's Lemma. Except for a short list of special cases, the actual calculation of any given hyperbolic metric is notoriously difficult.

By the formula

$$\rho_\Sigma(z) = h(|z|^2),$$

we obtain that the Gauss curvature is given by

$$K = \frac{4(|z|^2 h'^2 - |z|^2 h h'' - h h')}{h^4}.$$

Setting $t = |z|^2$, we obtain that

$$(6.1) \quad K = -\frac{1}{h^2} \left(\frac{4th'(t)}{h} \right)'.$$

As $K \leq 0$ it follows that

$$\left(\frac{4th'(t)}{h} \right)' \geq 0.$$

Therefore the function

$$\frac{4th'(t)}{h}$$

is increasing, i.e.

$$(6.2) \quad t \geq s \Rightarrow \frac{4th'(t)}{h(t)} \geq \frac{4sh'(s)}{h(s)},$$

and in particular

$$(6.3) \quad t \geq 0 \Rightarrow \frac{4th'(t)}{h(t)} \geq 0.$$

In this case we obtain that h is an increasing function.

The examples of hyperbolic surfaces are:

a) The Poincaré disk \mathbb{U} with the hyperbolic metric

$$\lambda = \frac{2}{1 - |z|^2}.$$

b) The punctured hyperbolic unit disk $\Delta = \mathbb{U} \setminus \{0\}$. The linear density of the hyperbolic metric on Δ is

$$\lambda_\Delta = \frac{1}{|z| \log \frac{1}{|z|}}.$$

c) The hyperbolic annulus $A(1/R, R)$, $R > 1$. The hyperbolic metric is given by

$$h_R(|z|^2) = \lambda_R(z) = \frac{\pi/2}{|z| \log R} \sec \left(\frac{\pi \log |z|}{2 \log R} \right).$$

In all these cases the Gauss curvature is $K = -1$.

6.2. Riemann metrics. In the case of the Riemann sphere, the Gauss-Bonnet theorem implies that a constant-curvature metric must have positive curvature K . It follows that the metric must be isometric to the sphere of radius $1/\sqrt{K}$ in \mathbb{R}^3 via stereographic projection.

d) In the z -chart on the Riemann sphere, the metric with $K = 1$ is given by

$$ds^2 = h_R^2(|z|^2) |dz|^2 = \frac{4|dz|^2}{(1 + |z|^2)^2}.$$

e) Another important case is Hamilton cigar soliton or in physics is known as *Wittens's black hole*. It is a Kähler metric defined on \mathbb{C} .

$$ds^2 = h^2(|z|^2) |dz|^2 = \frac{|dz|^2}{1 + |z|^2}.$$

The Gauss curvature is given by

$$K = \frac{2}{1 + |z|^2}.$$

In both these cases $K > 0$. This means that

$$\left(\frac{4th'(t)}{h} \right)' \leq 0.$$

Therefore the function

$$\frac{4th'(t)}{h}$$

is decreasing i.e.

$$(6.4) \quad t \geq s \Rightarrow \frac{4th'(t)}{h(t)} \leq \frac{4sh'(s)}{h(s)}.$$

In this case we obtain that h is a decreasing function.

REFERENCES

- [1] AHLFORS, L.; SARIO L.: *Riemann Surfaces (1st ed.)*, Princeton, New Jersey: Princeton University Press, p. **204** (1960).
- [2] ASTALA, K.; IWANIEC, T.; MARTIN, G.: *Deformations of annuli with smallest mean distortion*. Arch. Ration. Mech. Anal. **195** (2010), no. 3, 899–921.
- [3] ASTALA, K., IWANIEC, T., MARTIN, G.J.: *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*. Princeton University Press, Princeton, 2009.
- [4] ASTALA, K., IWANIEC, T., MARTIN, G., ONNINEN, J.: *Extremal mappings of finite distortion*. Proc. London Math. Soc. **91**(3), 655–702 (2005).
- [5] BUSER, P.: *Geometry and Spectra of Compact Riemann Surfaces*. Birkhäuser Verlag, Basel-Boston-New York, 1992.
- [6] BUSER, P.; SEPPÄLÄ, M.; SILHOL, R.: *Triangulations and moduli spaces of Riemann surfaces with group actions*. Manuscripta Math. **88** (1995), no. 2, 209–224.
- [7] CHOI, H.; TREIBERGS, A.: *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space*, J. Differential Geometry 32 (1990), 775–817.
- [8] HAN, Z.-C.: *Remarks on the geometric behavior of harmonic maps between surfaces*. Chow, Ben (ed.) et al., *Elliptic and parabolic methods in geometry*. Proceedings of a workshop, Minneapolis, MN, USA, May 23–27, 1994. Wellesley, MA: A K Peters. 57–66 (1996).
- [9] HENCL, S., KOSKELA, P., ONNINEN, J.: *A note on extremal mappings of finite distortion*. Math. Res. Lett. **12**, 231–237 (2005).
- [10] HENCL, S., KOSKELA, P.: *Regularity of the inverse of a planar Sobolev homeomorphism*. Arch. Ration. Mech. Anal. **180**, 75–95 (2006).
- [11] HILDEBRANDT, S., KAUL, H., WIDMAN, K.-O.: *An existence theorem for harmonic mappings of Riemannian manifolds*. Acta Math. **138** (1977), no. 1-2, 1–16.
- [12] IWANIEC, T., KOVALEV, L. K., ONNINEN, J.: *The Nitsche conjecture*, arXiv:0908.1253.
- [13] IWANIEC, T., KOVALEV, L. K., ONNINEN, J.: *Doubly connected minimal surfaces and extremal harmonic mappings*. arXiv:0912.3542
- [14] IWANIEC, T., ONNINEN, J.: *Hyperelastic deformations of smallest total energy*. Arch. Ration. Mech. Anal. **194** (2009), no. 3, 927–986.
- [15] JOST, J.: *Univalence of harmonic mappings between surfaces*. J. Reine Angew. Math. **324** (1981), 141–153.
- [16] KALAJ, D.: *Harmonic maps between annuli on Riemann surfaces*. To appear in Israel J. Math. (arXiv:1003.2744).
- [17] KALAJ, D.: *On the univalent solution of PDE $\Delta u = f$ between spherical annuli*. J. Math. Anal. Appl. **327** (2007), no. 1, 1–11.
- [18] KALAJ, D.: *On the Nitsche conjecture for harmonic mappings in \mathbb{R}^2 and \mathbb{R}^3* . Israel J. Math. **150** (2005), 241–251.
- [19] KALAJ, D.; MATELJEVIĆ, M.: *Inner estimate and quasiconformal harmonic maps between smooth domains*, J. Anal. Math. **100** (2006), 117–132.
- [20] LYZZAIK, A.: *The modulus of the image of annuli under univalent harmonic mappings and a conjecture of J. C. C. Nitsche*. J. London Math. soc., (2) **64** (2001), pp. 369–384.
- [21] MILNOR, T.: *Harmonic maps and classical surface theory in Minkowski 3-space*. Trans. Amer. Math. Soc. **280** (1983), no. 1, 161–185.
- [22] NITSCHKE, J.C.C.: *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly **69** (1962), 781–782.

- [23] RUH, E.; VILMS, J.: *The tension field of the Gauss map*. Trans. Amer. Math. Soc. **149** (1970) 569–573.
- [24] SCHOEN, R.; YAU, S. T.: *Lectures on harmonic maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997. vi+394 pp.
- [25] WAN, T.: *Constant mean curvature surface, harmonic maps, and universal Teichmüller space*. J. Differential Geom. 35 (1992), no. 3, 643–657.
- [26] WEITSMAN, A.: *Univalent harmonic mappings of annuli and a conjecture of J.C.C. Nitsche*. Israel J. Math. **124**, 327–331 (2001)

UNIVERSITY OF MONTENEGRO, FACULTY OF NATURAL SCIENCES AND MATHEMATICS,
CETINJSKI PUT B.B. 8100 PODGORICA, MONTENEGRO
E-mail address: davidk@t-com.me